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Joint Monitoring of Process Location and Dispersion Based on CUSUM Procedure and Generalized Likelihood Ratio in the Presence of Measurement Errors

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In this paper, we propose a new methodology based on the combination of cumulative sum procedure and generalized likelihood ratio statistic for joint monitoring of the process location and dispersion. Then, we explore the effect of measurement errors on detecting ability of the proposed control chart when (i) the variance of measurement error is constant (ii) the variance of measurement error increases linearly as the level of the process mean increases. We also utilize multiple measurements on each sample point in order to decrease the adverse effects of measurement errors on the performance of the proposed control charts. Two numerical examples based on simulation studies are given to evaluate the ability of the proposed methods in terms of average run length, median run length, standard deviation of run length, and the first and third quantile points of the run length distribution (Q_1 and Q_3). Finally, a real life example is given to illustrate the application of the proposed method. Copyright © 2017 John Wiley & Sons, Ltd.

Keywords: measurement errors; cumulative sum control chart; joint monitoring; multiple measurements

1. Introduction

The cumulative sum (CUSUM) control chart directly incorporates the information of both the current and preceding sample values by plotting the cumulative sum of the deviations of the sample values from a given target value (Montgomery[1]). Hence, CUSUM control charts are more sensitive to detect small and moderate shifts in the process location and dispersion in comparison with Shewhart – type control charts. Since the first CUSUM monitoring scheme was introduced by Page,[2] many researchers have proposed different modifications on CUSUM procedure to monitor the process location and dispersion in different statistical monitoring applications. In the rest of this section, we mention the recent CUSUM-based control charts which are proposed for monitoring the process location and dispersion:

Knoth[3] provided a methodology for computing the average run length (*ARL*) values for CUSUM schemes based on the sample variance S^2. Jiao and Helo[4] proposed an algorithm to an optimal design of a CUSUM control chart to detect mean shifts. Their proposed algorithm optimizes the sample size, sampling interval, control limit, and the reference parameter of the CUSUM control chart through minimizing the overall mean value of a Taguchi loss function over the probability distribution of the random process mean shift. Castagliola and Maravelakis[5] proposed a CUSUM control chart to monitor process dispersion with estimated process variance. They compared the performance of their proposed methodology with the same control chart in which the process parameters are assumed to be known. Shu *et al.*[6] developed an algorithm based on piecewise collocation method to compute the run-length distributions of CUSUM scale control charts. They proved that the proposed method provide more accurate approximation for the run-length distribution rather than the conventional Gauss-type quadrature-based method applied to the CUSUM location control charts. Huang *et al.*[7] introduced an algorithm based on the piecewise collocation method for computing the run-length distribution of CUSUM control chart in environments with skewed quality characteristics such as gamma distribution. Abujiya *et al.*[8] proposed a monitoring schemed based on the combination of Shewhart and CUSUM range R statistics to improve the

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performance of control chart to detect out-of-control situations in the process dispersion. Abujiya *et al.*[9] provided CUSUM control charts for monitoring the process dispersion with normal data based on the well-structured sampling techniques including extreme ranked set sampling, extreme double ranked set sampling and double extreme ranked set sampling. Su *et al.*[10] proposed a CUSUM control chart in which the probability density function of the in-control process distribution is not required to be known. They used kernel density estimation methods to estimate the density of an in-control process distribution.

The control charts for monitoring the process mean and the process variability have been usually developed separately in the literature. In practice, the control charts for monitoring both process mean and variability should be implemented together because the assignable causes can affect both of them. In recent years, quality practitioners have concentrated on joint monitoring of process mean and dispersion. In the following paragraph, we only focus on the recent researches provided for joint monitoring of the process location and dispersion under different processes:

Wu[11] proposed a real-time model based on the wavelet transform and probabilistic neural networks for the joint recognition of both mean and variance patterns. Chowdhury *et al.*[12] proposed a distribution-free Shewhart-type control chart based on the Cucconi statistic control chart. They also introduced a diagnostic procedure to determine the type of process shifts when the chart triggers an out-of-control signal. Prajapati and Singh[13] designed a modified joint $\overline{X}$ and R control chart based on the sum of chi-squares theory for simultaneous monitoring purpose. Maleki and Amiri[14] introduced a novel neural network-based model for simultaneous monitoring of the mean vector and covariance matrix under mixed multivariate-attribute quality characteristics. For detailed information, refer to review paper provided by McCracken and Chakraborti[15].

In most real production systems, the measuring instruments have some inherent sources of inaccuracy. As a result, the observed value of the quality characteristic under investigation does not equal to the accurate value. The difference between the observed value of the quality characteristic under investigation and the real one which is due to the measuring instruments or operators is called as 'measurement errors'. The measurement errors can seriously affect the ability of control charts in detecting various out-of-control conditions. The effect of measurement errors on different control charts are well documented in the literature. Here, we only concentrate on the most recent researches in this area.

Moameni *et al.*[16] investigated the effect of measurement errors on the effectiveness of the fuzzy control chart in detecting out-of-control situations using a linear covariate model. Abbasi[17] provided a comparison study among the performance of three control charts to monitor the process mean including Shewhart, CUSUM, and exponentially weighted moving average (EWMA) charts using the two component measurement error (TCME) model. Saghaei *et al.*[18] derived the cost function of an EWMA control chart considering the measurement errors and taking multiple measurements where the Taguchi loss functions is used to compute the imposed cost due to the poor quality products. They presented a numerical example and computed the *ARL* values using a Markov chain method. They also obtained the optimal values of the chart parameters using a genetic algorithm. Noorossana and Zerehsaz[19] explored the effect of the classical additive measurement errors model on the most commonly used control charts for monitoring simple linear profiles in Phase II. Abbasi[20] studied the effect of the TCME on the EWMA control chart. They also provided a cost function analysis to determine the optimal number of multiple measurements and the sample size to reduce the effect of TCME. Khati Dizabadi *et al.*[21] investigated the effect of measurement errors with linearly increasing-type variance on the performance of maximum exponentially weighted moving average and mean-squared deviation control chart to detect out-of-control scenarios. The effect of contaminated data due to the gauge measurement errors on the performance of ELR control chart under ranked set sampling procedure is explored by Ghashghaei *et al.*[22].

In this paper, we propose a new methodology based on the CUSUM procedure and generalized likelihood ratio (GLR) for the joint monitoring of the process location and dispersion. Then, we study the effect of measurement errors in two scenarios: (i) constant variance, (ii) linearly increasing-type variance. We also utilize multiple measurements at each sample point as a remedial approach to decrease the adverse effect of measurement errors on the ability of the proposed control chart. The rest of the paper is organized as follows: Cumulative sum control chart for monitoring the process mean and the process variability are presented in Sections 2.1 and 2.2, respectively. In Section 2.3, the proposed method for the joint monitoring of the process mean and variability is presented. Effect of measurement errors for both constant variance and linearly increasing variance are explained in Sections 3.1 and 3.2, respectively. In order to decrease the measurement errors effect, taking several measurements on each sample point is suggested in Section 4. In Section 5, the performance of the proposed method is evaluated using two numerical examples based on simulation studies. In Section 6, a real life example is given to illustrate the application of the proposed method. Finally, some concluding remarks and suggestions for future research are provided in Section 7.

2. Cumulative sum control chart

2.1. Cumulative sum control chart for monitoring process mean

Let X_t be the tth observation of the quality characteristic of interest. When the process is in-control, X_t is normally distributed with mean μ_0 and variance σ_0^2. For tth; $t = 1, 2, \ldots$ sample, the control chart statistics to detect increasing and decreasing mean shifts are obtained according to Eqs (1) and (2), respectively:

$$A_t^+ = \max\{0, X_t - (\mu_0 + K) + A_{t-1}^+\}, \tag{1}$$

$$A_t^- = \max\{0, (\mu_0 - K) - X_t + A_{t-1}^-\}, \tag{2}$$

where $A_t^+ = A_t^- = 0$ and K is called as the reference value of the control chart. Note that, the reference value usually is selected equal to $|\mu_1 - \mu_0|/2$, where μ_1 is the out-of-control value of the process mean that we aim to detect quickly. For tth; $t = 1, 2, \ldots$ sample, the CUSUM statistic to detect both increasing and decreasing mean shifts is given as follows:

$$A_t = \max\{A_t^+, A_t^-\} \tag{3}$$

The chart triggers an out-of-control signal when $A_t > H_1$, where H_1 is determined such that the in-control average run length (ARL_0) value of the control chart is equal to a pre-specified constant.

2.2. *Cumulative sum control chart for monitoring process dispersion*

For tth; $t = 1, 2, \ldots$ sample, the chart statistic to detect increasing variance shifts is calculated as follows:

$$B_t^+ = \max\{0, B_{t-1}^+ + (X_t^2 - \lambda^+)\} \tag{4}$$

where $B_0^+ = 0$ and λ^+ is the reference value according to the following formula:

$$\lambda^+ = \frac{\sigma_+^2 \times \ln(\sigma_+^2)}{\sigma_+^2 - 1} > 0 \tag{5}$$

Similarly, the chart statistic to detect decreasing variance shifts corresponding to tth sample can also be calculated as follows:

$$B_t^- = \max\{0, B_{t-1}^- + (\lambda^- - X_t^2)\} \tag{6}$$

where $B_0^- = 0$ and λ^- is the reference value which can be determined as follows:

$$\lambda^- = \frac{\sigma_-^2 \times \ln(\sigma_-^2)}{\sigma_-^2 - 1} > 0 \tag{7}$$

For tth sample, the statistic of the CUSUM chart to detect both increasing and decreasing variance shifts can be calculated as follows:

$$B_t = \max\{B_t^+, B_t^-\} \tag{8}$$

The chart triggers an out-of-control signal when $B_t > H_2$ where H_2 is determined such that the value of ARL_0 is equal to some pre-specified constant.

2.3. *Proposed method for joint monitoring the process mean and variability*

Joint monitoring of the process mean and variability is equivalent to considering the following hypothesis:

$$\begin{cases} H_0 : \mu = \mu_0 \text{ and } \sigma^2 = \sigma_0^2 \\ H_1 : \mu \neq \mu_0 \text{ or } \sigma^2 \neq \sigma_0^2 \end{cases} \tag{9}$$

In order to decide between H_0 and H_1, the GLR statistic corresponding to tth sample (denoted by LR_t) can be utilized as follows:

$$LR_t = \overline{X}_t^2 + S_t^2 - \ln(S_t^2), \tag{10}$$

where $\overline{X}_t = \sum_{j=1}^{n} X_{tj}/n$ and $S_t^2 = \sum_{j=1}^{n} (x_{tj} - \overline{X})_t/n$ are the sample mean and the sample variance of tth sample. To increase the sensitivity of the control chart in detecting small and moderate shifts, we incorporate the CUSUM procedure in constructing the LR_t statistic. Hence, the monitoring statistic concerning the joint monitoring of the process location and dispersion will be

$$CLR_t = A_t^2 + B_t - \ln(B_t) \tag{11}$$

The proposed joint monitoring scheme triggers an out-of-control signal when $CLR_t > H$, where the value of H is determined such that the ARL_0 value is equal to some pre-determined constant.

3. Effect of measurement errors

In this section, the effect of measurement errors on the ability of the proposed control chart for joint monitoring of the process location and dispersion is described.

3.1. *Effect of measurement errors in the case of constant variance*

In order to incorporate the measurement errors into the proposed control chart, we assume the following covariate model:

$$Y_t = a + bX_t + \varepsilon_t, \tag{12}$$

where Y_t (linearly related to X_t) is the observed value of tth observation. Here, ε_t is the random error variable independent of X_t and is normally distributed with mean zero and constant variance of σ_ε^2. In addition a and b are defined as the intercept and slope constants, respectively. As the result Y follows normal distribution as follows:

$$Y \sim N\left(a + b\mu_X, b^2\sigma_X^2 + \sigma_\varepsilon^2\right) \tag{13}$$

For tth observation, the extended chart statistic to detect increasing and decreasing mean shifts in the presence of measurement errors with the constant variance are computed as follows:

$$A_t^{+\prime} = \max\left\{0, Y_t - (\mu_0 + K) + A_{t-1}^{+}{}'\right\} \tag{14}$$

$$A_t^{-\prime} = \max\left\{0, (\mu_0 - K) - Y_t + A_{t-1}^{-}{}'\right\} \tag{15}$$

where $A_t^{+\prime} = A_t^{-\prime} = 0$. Then the CUSUM statistic for detecting mean shifts in the presence of error term with a constant variance will be

$$A_t^{'} = \max\left\{A_t^{+\prime}, A_t^{-\prime}\right\} \tag{16}$$

For tth sample, the extended statistic to detect increasing and decreasing variance shifts in the case of constant variance is obtained based on Eqs (17) and (18), respectively:

$$B_t^{+\prime} = \max\left\{0, B_{t-1}^{+} + \left(Y_t^2 - \lambda^+\right)\right\} \tag{17}$$

$$B_t^{-\prime} = \max\left\{0, B_{t-1}^{-} + \left(\lambda^- - Y_t^2\right)\right\} \tag{18}$$

where $B_0^{+\prime} = B_0^{-\prime} = 0$. Note that λ^+ and λ^- are calculated using Eqs (5) and (7), respectively. Finally the extended CUSUM statistic to monitor both location and dispersion parameters under measurement errors with constant variance can be calculated as follows:

$$B_t^{'} = \max\left\{B_t^{+\prime}, B_t^{-\prime}\right\} \tag{19}$$

After constructing the CUSUM statistics for monitoring process location and dispersion, the extended statistic for joint monitoring purpose in the presence of measurement errors is defined as follows:

$$CLR_t^{'} = A_t^{'2} + B_t^{'} - \ln\left(B_t^{'}\right) \tag{20}$$

The proposed control chart triggers an out-of-control signal when $CLR_t^{'} > H^{'}$, where $H^{'}$ is determined based on simulation experiments to have ARL_0 equal to a pre-specified constant.

3.2. *Effect of measurement errors in the case of linearly increasing variance*

In some production systems, the assumption of constant variance for measurement errors is violated, and the variance of error term increases linearly as the level of the process mean increases. To model such situations, we consider the covariate model similar to Section 3.1 according to Eq. (12). All assumptions are considered similar to previous subsection except the one that variance of ε_t is proportional to μ_X and is equal to $C + D\mu_X$ where C and D are two constant values. Hence, Z is a normally distributed quality characteristic with the following parameters:

$$Z \sim N\left(a + b\mu_X, b^2\sigma_X^2 + C + D\mu_X\right) \tag{21}$$

For tth sample, Eqs (1) and (2) are written as follows:

$$A_t^{+''} = \max\left\{0, Z_t - (\mu_0 + K) + A_{t-1}^{+}{}''\right\} \tag{22}$$

$$A_t^{-''} = \max\left\{0, (\mu_0 - K) - Z_t + A_{t-1}^{-}{}''\right\} \tag{23}$$

where $A_t^{+''} = A_t^{-''} = 0$. Then, Eq. (3) for joint monitoring process location in the presence of measurement errors with linearly increasing variance is rewritten as follows:

$$A_t^{''} = \max\left\{A_t^{+''}, A_t^{-''}\right\} \tag{24}$$

For tth sample, Eqs (4) and (6) in the case of linearly increasing variance for error term are rewritten as follows:

$$B_t^{+''} = \max\left\{0, B_{t-1}^{+}{}'' + \left(Z_t^2 - \lambda^+\right)\right\}, \tag{25}$$

$$B_t^{-''} = \max\left\{0, B_{t-1}^{-}{}'' + \left(\lambda^- - Z_t^2\right)\right\}, \tag{26}$$

where $B_0^{+''} = B_0^{-''} = 0$. Similarly, λ^+ and λ^- are calculated using Eqs (5) and (7), respectively. Finally we have

$$B_t^{''} = \max\left\{B_t^{+''}, B_t^{-''}\right\} \tag{27}$$

The proposed CUSUM statistic for joint monitoring of process location and dispersion in the presence of measurement error with linearly increasing variance is

$$CLR_t^{''} = A''^2_t + B_t^{''} - \ln\left(B_t^{''}\right) \tag{28}$$

The proposed control chart triggers an out-of-control signal when $CLR_t^{''} > H^{''}$, where $H^{''}$ is determined based on simulation experiments to have ARL_0 equal to a pre-specified number.

4. Multiple measurements

4.1. Multiple measurements under constant error variance

In this subsection, we incorporate multiple measurements approach into the proposed control chart when the constant variance is considered. Taking k measurements on each sample point, the covariate model can be rewritten according to Eq. (29):

$$Y_{tj} = a + bX_t + \varepsilon_{tj}. \tag{29}$$

For tth sample, $\overline{Y}_t = \frac{1}{k}\sum_{j=1}^{k} Y_{tj}$ where $\overline{Y}_t \sim N\left(a + b\mu_X, b^2\sigma_X^2 + \frac{\sigma_\varepsilon^2}{k}\right)$. In order to derive the monitoring statistic while multiple measurements approach is utilized, we substitute $\overline{Y}_t$ instead of Y_t in all equations of Section 3.1. For $t = 1, 2, \ldots$, we denote the control statistic by $MCLR_t^{'}$ which can be determined as follows:

$$MCLR_t^{'} = MA'^2_t + MB_t^{'} - \ln\left(MB_t^{'}\right) \tag{30}$$

Note that $MA_t^{'}$ and $MB_t^{'}$ are obtained with replacing Y_t by $\overline{Y}_t$ in Eqs (14)–(19). The proposed control chart triggers an out-of-control signal when $MCLR_t^{'} > MH^{'}$ where $MH^{'}$ is set such that ARL_0 be a pre-specified number.

4.2. Multiple measurements under constant error variance

Utilizing multiple measurements on each sample point in the case of linearly increasing variance for error term is discussed in this subsection. The covariate model in this case when k units are measured on each sample point is similar to Section 4.1. Similarly, for $t = 1, 2, \ldots$ sample point, we have $\overline{Z}_t = \frac{1}{k}\sum_{j=1}^{k} Z_{tj}$ where $\overline{Z}_t \sim N\left(a + b\mu_X, b^2\sigma_X^2 + \frac{C + D\mu_X}{k}\right)$. When multiple measurements are applied, the chart statistic for joint monitoring of the process location and dispersion in the presence of measurement error with linearly increasing variance can be written as follows:

$$MCLR_t^{''} = MA''^2_t + MB_t^{''} - \ln\left(MB_t^{''}\right) \tag{31}$$

In Eq. (31), $MA_t^{''}$ and $MB_t^{''}$ are obtained with replacing Z_t by $\overline{Z}_t$ in Eqs (22)–(27). The proposed control chart triggers an out-of-control signal when $MCLR_t^{''} > MH^{''}$, where $MH^{''}$ is set such that the value of ARL_0 be a pre-specified number.

Table I. RL characteristics of the proposed CLR chart against LR when $ARL_0 = 200$

chart	(μ_1, σ_1)	(0,1)	(0.25,1)	(0.5,1)	(1,1)	(0,1.25)	(0,1.5)	(0,2)	(0.25,1.25)
CLR	ARL	199.7517	85.4564	28.7049	8.5071	34.2370	15.5350	7.1773	27.3861
	SDRL	234.3513	97.0357	28.2774	5.8742	32.2712	13.1843	5.5153	25.6388
	MRL	124	55	21	8	26	13	6	21
	Q_1	27	15	8	5	10	6	3	8
	Q_3	287	124	40	11	49	22	10	40
GLR	ARL	200.4887	161.4730	98.1894	31.0791	43.9163	17.5499	6.5440	38.6787
	SDRL	199.1889	160.9186	98.2398	30.6248	42.8893	17.2166	5.9840	38.4654
	MRL	140.5	113	68	22	31	12	5	27
	Q_1	58	47	29	9	13	5	2	11
	Q_3	278	224	136	43	60	24	9	53
chart	(μ_1, σ_1)	(0.25,1.5)	(0.25,2)	(0.5,1.25)	(0.5,1.5)	(0.5,2)	(1,1.25)	(1,1.5)	(1,2)
CLR	ARL	14.4036	7.0113	17.3121	11.5437	6.4807	7.7543	6.8586	5.1821
	SDRL	12.1344	5.3789	15.4819	9.4711	4.9823	5.6764	5.0548	3.8182
	MRL	12	6	13	9	5	7	6	4
	Q_1	5	3	6	5	3	4	3	2
	Q_3	20	10	24	16	9	10	9	7
GLR	ARL	16.5450	6.3816	29.5986	14.0628	6.0589	14.0601	8.8253	4.9151
	SDRL	16.1045	5.8584	29.2305	13.5703	5.4211	13.4785	8.1921	4.3520
	MRL	12	5	21	10	4	10	6	4
	Q_1	5	2	9	4	2	4	3	2
	Q_3	23	9	41	19	8	19	12	7

ARL, average run length; MRL, median run length; RL, run length; SDRL, standard deviation of run length.

Table II. Characteristics of the CLR chart under constant error variance when a = 0, b = 1

σ_e^2	(μ_1,σ_1)	(0,1)	(0.25,1)	(0.5,1)	(1,1)	(0,1.25)	(0,1.5)	(0,2)	(0.25,1.25)
0.1	ARL	200.1959	92.2232	30.8441	9.3805	37.8983	17.9281	8.2264	30.3037
	SDRL	230.2320	102.6566	30.3461	6.4086	34.0604	14.5256	6.1893	27.2573
	MRL	126	61	23	8	31	15	7	24
	Q_1	29	17	9	5	11	7	4	9
	Q_3	283	132	43	13	55	26	11	44
0.3	ARL	199.1240	105.7217	39.5620	11.8952	49.7032	25.3991	11.8284	40.3610
	SDRL	197.0791	104.3024	36.9302	7.9514	38.8114	18.7001	8.3578	32.2669
	MRL	155	80	30	11	45	23	10	35
	Q_1	46	25	13	7	19	11	6	15
	Q_3	291	155	56	16	73	37	17	59
0.5	ARL	200.1111	129.9099	52.7241	15.7331	72.1386	39.2750	18.2976	58.3810
	SDRL	147.1979	105.8964	45.9071	10.0662	48.5850	25.9344	11.9811	42.1617
	MRL	196	118	42	15	72	38	17	56
	Q_1	87	41	19	9	35	19	9	24
	Q_3	294	195	76	21	105	57	26	87
σ_e^2	(μ_1,σ_1)	(0.25,1.5)	(0.25,2)	(0.5,1.25)	(0.5,1.5)	(0.5,2)	(1,1.25)	(1,1.5)	(1,2)
0.1	ARL	16.2151	8.0594	19.3429	12.9646	7.4584	8.6344	7.6551	5.8864
	SDRL	13.3230	6.0918	17.0296	10.5292	5.6073	6.2204	5.5899	4.2463
	MRL	13	7	15	11	6	7	7	5
	Q_1	6	3	7	5	3	4	4	3
	Q_3	23	11	27	18	10	12	10	8
0.3	ARL	22.8068	11.4541	25.5487	15.5698	9.9340	10.6486	9.6087	7.5857
	SDRL	17.1992	8.1030	21.0647	10.1325	7.0589	7.5145	6.8240	5.2854
	MRL	20	10	21	14	9	9	8	7
	Q_1	9	5	10	9	5	5	5	4
	Q_3	33	16	37	21	14	14	13	10
0.5	ARL	34.8121	17.4527	36.0500	26.1058	15.5815	14.5577	13.3724	10.8127
	SDRL	24.0709	11.6540	28.2451	18.8666	10.6248	9.6374	9.1174	7.3004
	MRL	33	16	31	23	14	13	12	9
	Q_1	16	9	15	12	8	8	7	6
	Q_3	51	24	52	37	22	20	18	15

ARL, average run length; MRL, median run length; SDRL, standard deviation of run length.

Table III. Characteristics of the CLR chart under constant variance error for different values of *k* when a = 0, b = 1, $\sigma_\varepsilon^2 = 0.5$

k	(μ_1,σ_1)	(0,1)	(0.25,1)	(0.5,1)	(1,1)	(0,1.25)	(0,1.5)	(0,2)	(0.25,1.25)
1	ARL	200.1111	129.9099	52.7241	15.7331	72.1386	39.2750	18.2976	58.3810
	SDRL	147.1979	105.8964	45.9071	10.0662	48.5850	25.9344	11.9811	42.1617
	MRL	196	118	42	15	72	38	17	56
	Q_1	87	41	19	9	35	19	9	24
	Q_3	294	195	76	21	105	57	26	87
3	ARL	200.4555	94.7238	32.9618	10.0584	40.2175	19.6556	9.2002	32.6417
	SDRL	223.4437	103.0600	31.6573	6.9308	34.4005	15.5078	6.7813	28.5764
	MRL	133	65	25	9	34	17	8	27
	Q_1	33	19	10	5	13	8	4	10
	Q_3	292	135	46	14	59	28	13	48
5	ARL	200.3020	91.4965	30.7813	9.3821	37.3332	17.6245	8.2259	29.9535
	SDRL	229.1546	101.2662	30.3739	6.4186	33.3802	14.3310	6.1717	27.0825
	MRL	127	61	23	8	30	15	7	23
	Q_1	30	17	9	5	11	7	4	9
	Q_3	285	131	43	13	55	25	11	44
k	(μ_1,σ_1)	(0.25,1.5)	(0.25,2)	(0.5,1.25)	(0.5,1.5)	(0.5,2)	(1,1.25)	(1,1.5)	(1,2)
1	ARL	34.8121	17.4527	36.0500	26.1058	15.5815	14.5577	13.3724	10.8127
	SDRL	24.0709	11.6540	28.2451	18.8666	10.6248	9.6374	9.1174	7.3004
	MRL	33	16	31	23	14	13	12	9
	Q_1	16	9	15	12	8	8	7	6
	Q_3	51	24	52	37	22	20	18	15
3	ARL	18.1696	8.9637	20.9270	14.2195	8.1277	9.1962	8.2620	6.4311
	SDRL	14.2482	6.5614	18.1227	11.2091	5.9301	6.5004	5.9940	4.5649
	MRL	15	8	17	12	7	8	7	5
	Q_1	7	4	8	6	4	5	4	3
	Q_3	26	13	30	20	11	12	11	9
5	ARL	16.1801	8.0420	19.1903	13.1223	7.4170	8.5868	7.6243	5.8737
	SDRL	13.2550	5.9927	16.7468	10.5459	5.5514	6.2278	5.5949	4.2318
	MRL	13	7	15	11	6	7	6	5
	Q_1	6	4	7	5	3	4	4	3
	Q_3	23	11	27	18	10	12	10	8

ARL, average run length; MRL, median run length; SDRL, standard deviation of run length.

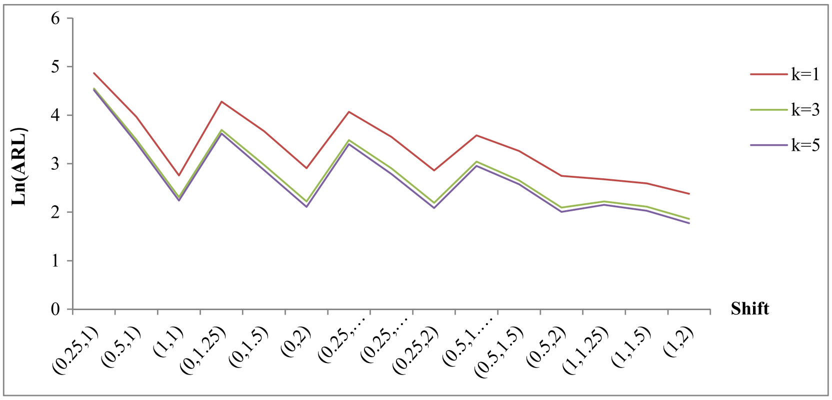

Figure 1. Comparison study under constant variance of error term. ARL, verage run length [Colour figure can be viewed at wileyonlinelibrary.com]

Table IV. The values of SSE under constant variance of error term

	$k = 1$	$k = 3$	$k = 5$
SSE	6981.652	237.415	76.07567

5. Performance evaluation

In this section, two numerical examples based on simulation studies are given in order to explore the ability of the proposed control charts in terms of some run length (RL)-based criteria namely ARL, median run length (MRL), standard deviation of run length (SDRL) and the first and third quantile points of the run length distribution (Q_1 and Q_3). In numerical example 1, firstly the ability of proposed CUSUM-based chart for joint monitoring purpose is investigated and compared with generalized likelihood ration statistic (denoted by LR). Then, the effect of measurement errors with constant variance on detecting ability of the proposed control chart is explored. After that, the results of utilizing multiple measurements in the case of measurement error with constant variance are studied. In numerical example 2, we consider linearly increasing variance for the error term and investigate the effect of measurement errors on the proposed control chart. Then, we explore multiple measurements to improve the detecting ability of control chart. Note that, in all simulation experiments, we set the upper control limit (*UCL*) of the proposed control charts to have $ARL_0 = 200$. Then, we evaluate the ability of proposed CUSUM-based charts to detect mean shifts, variance shifts and the joint shifts in the both under constant and linearly increasing variance for measurement error. It is notable that the proposed monitoring schemes can be used when individual observations are used for joint monitoring the process location and dispersion. Hence, the sample size of $n = 1$ is used in all simulation experiments. Note that, we use MATLAB computer packaged in our simulation experiments.

5.1. Example 1

Assume that the interested quality characteristic follows the standardized normal distribution ($X \sim N(0, 1)$) and the error term which is independent form X is a normally distributed variable with mean zero and constant variance of σ_ε^2. Also assume that the covariate model parameters are $a = 0$ and $b = 1$. Table I contains the results of the proposed CLR control chart (which is obtained by incorporating the CUSUM procedure into GLR statistic) against the *GLR* to detect various out-of-control scenarios in terms of ARL, SDRL, MRL, and Q_1 and Q_3based on 20,000 simulation experiments. Table I shows that the proposed *CLR* method outperforms GLR statistic to detect most step shifts in terms of all *RL*-based criteria considered.

In Table II, the RL features of the proposed control chart under covariate model under different values of σ_ε^2 are evaluated. It is clear that in each shift type considered including mean shifts, variance sifts, and joint shifts, the detecting ability of the control chart decrease as the variance of measurement error increases. As the result, we can conclude that the measurement error with constant variance can adversely affect the detecting ability of the proposed *CLR* control chart.

In Table III, the RL features of the proposed chart are assessed when multiple measurements approach is utilized by considering the constant value for the measurement error variance. It is seen that, as the number of measurements for each sample point increases, all the RL-based criteria tend to decrease. Consequently, the multiple measurements error can cover the effect of measurement errors with constant variance. Although, increasing the number of measurements on each sample point leads to improvement in statistical features of the *CLR* control chart; however, the cost of sampling will increase. As the result, there is a trade-off between statistical and economical objectives which can be decided by the process practitioners. Figure 1 graphically illustrates the effect of measurement errors with constant variance and utilizing multiple measurements on each unit. We also provide a comparison study on the effect of parameter k on the detecting ability of control chart in terms of *SSE* (Sum of Squares due to Error) in Table IV. It is seen that as the value of k increases, the value of *SSE* decreases exponentially.

Table V. Characteristics of the CLR chart under linearly increasing variance error for different values of D and $C = 0$

D	(μ_1,σ_1)	(0.5,1)	(1,1)	(1.5,1)	(2,1)	(0.5,1. 5)	(0.5,2)	(0.5,2.5)	(1,1.5)
1	ARL	200.7490	37.0081	14.4341	7.8678	46.2526	21.7217	13.1054	23.5133
	SDRL	154.1162	28.3462	9.4856	4.9283	32.0941	14.6395	8.7504	16.7913
	MRL	210	33	13	8	47	21	12	21
	Q_1	30	15	8	6	20	10	7	11
	Q_3	307	55	19	11	69	32	18	34
3	ARL	199.9733	52.5543	23.0900	14.2413	97.9245	53.7024	33.7647	40.8737
	SDRL	111.3061	34.9502	14.6902	8.3479	54.9027	30.7573	19.6857	26.2358
	MRL	233	50	21	13	110	56	34	39
	Q_1	154	26	13	9	64	32	20	21
	Q_3	278	77	31	19	138	77	48	59
5	ARL	199.0078	58.4945	27.5284	17.4220	121.2433	75.6741	49.7395	48.5224
	SDRL	100.8839	36.2279	17.0420	10.1730	63.3640	40.6218	27.5064	29.4549
	MRL	230	56	25	16	137	81	51	47
	Q_1	165	32	16	11	87	49	30	27
	Q_3	268	84	37	23	167	106	70	69
D	(μ_1,σ_1)	(1,2)	(1,2.5)	(1.5,1.5)	(1.5,2)	(1.5,2.5)	(2,1.5)	(2,2)	(2,2.5)
1	ARL	15.6419	11.0810	12.5352	10.5078	8.6768	8.2811	7.5038	6.7376
	SDRL	10.6965	7.3574	8.4683	7.1041	5.7914	5.0551	4.7261	4.3135
	MRL	14	10	11	9	8	7	7	6
	Q_1	8	6	7	5	4	5	4	4
	Q_3	22	15	17	14	12	11	10	9
3	ARL	30.9458	23.4974	21.0685	18.7970	16.3489	13.6910	12.9266	11.9571
	SDRL	19.0463	14.1503	13.3347	11.8188	9.9935	8.0794	7.8043	7.1357
	MRL	29	22	19	17	15	12	12	11
	Q_1	17	13	12	10	9	8	8	7
	Q_3	44	33	29	26	22	18	17	16
5	ARL	39.2674	31.5049	25.6554	23.0275	20.9157	16.9279	16.0096	14.9621
	SDRL	23.2068	18.3371	15.6013	13.9582	12.4158	9.8346	9.4148	8.7291
	MRL	38	30	23	21	19	15	15	14
	Q_1	22	18	15	13	12	10	10	9
	Q_3	55	44	35	31	28	22	21	20

ARL, average run length; MRL, median run length; SDRL, standard deviation of run length.

Table VI. Characteristics of the CLR chart under linearly increasing variance error for different values of C and D = 1

C	(μ_1,σ_1)	(0.5,1)	(1,1)	(1.5,1)	(2,1)	(0.5,1. 5)	(0.5,2)	(0.5,2.5)	(1,1.5)
1	ARL	200.2115	69.3081	26.5693	15.7745	99.3765	54.3510	33.9643	49.9606
	SDRL	113.7497	48.2776	16.7086	8.2819	55.2427	31.2795	19.7070	32.9482
	MRL	234	65	25	15	111	57	34	47
	Q_1	148	33	16	11	64	32	20	25
	Q_3	280	103	36	20	140	77	48	73
2	ARL	199.5296	86.7449	34.5239	20.6214	122.5092	76.7545	50.4553	66.8594
	SDRL	101.9144	56.2815	21.0897	10.7315	63.3957	40.5700	27.6435	42.1671
	MRL	231	84	32	20	138	82	52	64
	Q_1	163	45	21	14	90	51	31	36
	Q_3	270	127	46	27	168	107	71	97
3	ARL	199.8333	96.9663	41.2656	24.4241	137.7040	92.3817	64.4592	78.7037
	SDRL	96.3625	61.0931	24.9494	12.7243	68.1991	47.5505	33.6528	48.4180
	MRL	229	93	39	24	155	100	67	76
	Q_1	168	52	26	17	103	62	41	43
	Q_3	266	141	55	31	187	128	89	114
C	(μ_1,σ_1)	(1,2)	(1,2.5)	(1.5,1.5)	(1.5,2)	(1.5,2.5)	(2,1.5)	(2,2)	(2,2.5)
1	ARL	35.7627	26.5316	24.2483	21.6206	18.5763	15.2029	14.4341	13.3508
	SDRL	22.7213	16.1586	15.4662	13.7210	11.6278	8.3873	8.3397	7.9879
	MRL	34	25	22	20	17	14	13	12
	Q_1	19	15	14	12	10	10	9	8
	Q_3	51	37	33	29	25	20	19	18
2	ARL	50.7350	38.4575	32.3361	29.1576	25.9315	19.7014	19.0119	17.8844
	SDRL	30.7790	22.8801	19.9900	18.1333	15.7642	10.6682	10.8201	10.3071
	MRL	49	37	30	27	24	19	18	16
	Q_1	28	22	19	17	15	13	12	11
	Q_3	73	54	43	39	35	25	25	23
3	ARL	61.9383	48.6568	38.2373	35.5598	31.7120	23.8068	22.9364	21.8265
	SDRL	36.9074	27.8224	23.7631	21.7618	19.1500	12.7148	12.6725	12.3921
	MRL	60	47	35	33	29	23	21	20
	Q_1	35	29	23	21	19	16	15	14
	Q_3	88	68	51	48	43	31	30	29

5.2. Example 2

Here, we assume that when the process is in-control, $X \sim N(0.5, 1)$. The error term which is independent form X is a normally distributed variable with mean zero and variance $C + D\mu_X$. Note that, similar to Section 5.1, the covariate model parameters are $a = 0$ and $b = 1$.

First the effect of parameter D is evaluated, and the results based on 20,000 simulation experiments are summarized in Table V. It is observed that as parameter D increase, the ability of control chart in detecting all out-of-control scenarios decreases. It is worth mentioning here that in comparison with error term with constant variance, the adverse effect of measurement error with linearly increasing variance on *CLR* control chart is more significant.

In Table VI, we give the results of sensitivity analysis on parameter C under different out-of-control step shifts in the case of linearly increasing variance. The results represent that in each step shift considered, increasing the parameter C leads to larger *RL*-based quantities.

Table VII displays the RL-based quantities when the multiple measurements approach is utilized when linearly increasing variance is considered for the measurement error. We see that the multiple measurements on each unit reduce the adverse effect of error with linearly increasing variance. It is also concluded that increasing the number of measurements on each sample point improves the detecting ability of the proposed control chart. See also Figure 2 in order to have an illustrative view on the effect of parameter k on the proposed control chart. A comparison study is also provided in Table VIII for investigating the effect of parameter k on the detecting ability of control chart in terms of *SSE*.

6. Case study

In this section, the application of our proposed monitoring scheme is illustrated by a real data set from a long-standing research project in the ambulatory monitoring system (Hawkins and Maboudou-Tchao[23]). In this work, subjects were equipped with

Table VII. Characteristics of the CLR chart under linearly increasing variance error for different values of k when C = 1, D = 1

k	(μ_1,σ_1)	(0.5,1)	(1,1)	(1.5,1)	(2,1)	(0.5,1. 5)	(0.5,2)	(0.5,2.5)	(1,1.5)
2	ARL	199.7268	53.3400	18.9617	11.1384	63.0666	30.8609	18.6448	33.3830
	SDRL	132.8063	40.4146	12.0612	5.7360	40.0541	19.5766	11.7275	23.5231
	MRL	224	48	18	11	67	31	18	31
	Q_1	86	22	11	8	33	16	10	15
	Q_3	295	80	26	14	92	45	26	49
3	ARL	199.2087	43.5470	15.3292	9.1119	46.5605	21.6801	13.1713	25.8253
	SDRL	154.7678	34.5982	9.9700	4.6348	32.1689	14.6307	8.6521	18.8107
	MRL	206	38	14	9	47	21	12	23
	Q_1	27	16	9	6	20	11	7	11
	Q_3	304	65	21	12	69	31	19	38
5	ARL	200.5342	34.2102	12.2015	7.2638	32.0810	14.8202	9.0351	19.0220
	SDRL	197.5562	28.9336	8.0673	3.6670	23.9455	10.5123	6.2648	14.4074
	MRL	165	29	11	7	30	13	8	16
	Q_1	13	11	7	5	12	7	4	8
	Q_3	308	50	17	9	48	21	13	28
k	(μ_1,σ_1)	(1,2)	(1,2.5)	(1.5,1.5)	(1.5,2)	(1.5,2.5)	(2,1.5)	(2,2)	(2,2.5)
2	ARL	22.4562	15.7926	16.7905	14.3188	11.8763	10.6985	9.9033	8.9737
	SDRL	14.9331	10.1294	10.9926	9.4770	7.7455	6.0231	6.0332	5.5261
	MRL	21	15	15	13	11	10	9	8
	Q_1	11	8	9	8	6	7	6	5
	Q_3	32	22	23	20	16	14	13	12
3	ARL	16.7313	11.6078	13.6016	11.1739	9.1209	8.5858	8.0062	7.0986
	SDRL	11.5164	7.6430	9,0290	7.5512	6.1025	4.9514	5.0193	4.5594
	MRL	15	10	12	10	8	8	7	6
	Q_1	8	6	7	6	5	5	5	4
	Q_3	24	16	18	15	12	11	11	9
5	ARL	11.9037	8.2156	10.4926	8.5239	6.7969	6.8483	6.1648	5.4845
	SDRL	8.4641	5.7176	7.2418	5.9195	4.6316	4.0182	3.9499	3.6370
	MRL	11	7	9	7	6	6	5	5
	Q_1	5	4	5	4	3	4	3	3
	Q_3	17	11	14	12	9	9	8	7

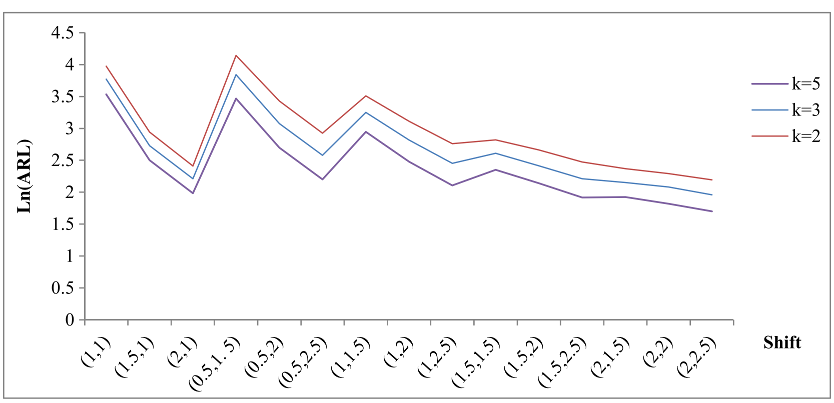

Figure 2. Comparison study under linearly increasing variance of error term [Colour figure can be viewed at wileyonlinelibrary.com]

Table VIII. The values of *SSE* under linearly increasing variance of error term

	k = 2	k = 3	k = 5
SSE	4222.518	1568.039	308.650

Table IX. The values of mean diastolic blood pressure

Sample number	DBP	Sample number	DBP
1	78.357	13	75.246
2	79.283	14	78.153
3	80.756	15	76.428
4	81.412	16	75.742
5	81.294	17	75.994
6	79.634	18	73.581
7	81.060	19	73.946
8	77.676	20	79.588
9	78.729	21	77.350
10	78.731	22	75.729
11	78.267	23	77.828
12	76.632	24	76.389

DBP, diastolic blood pressure.

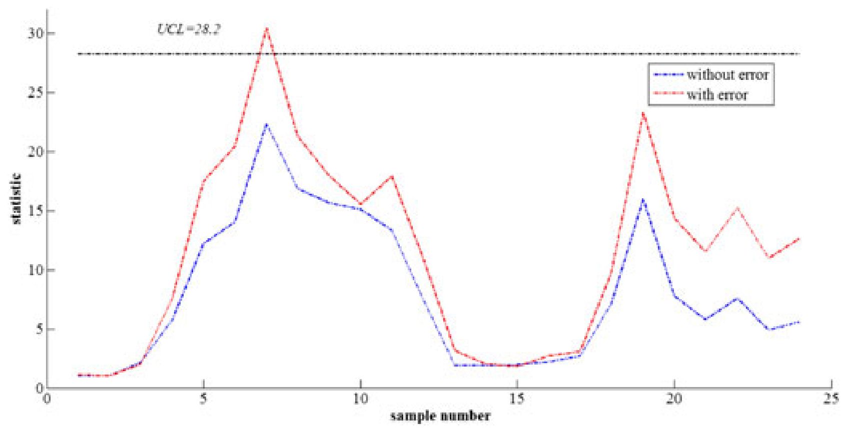

Figure 3. The effect of measurement error on the rate of false alarm. UCL, upper control limit [Colour figure can be viewed at wileyonlinelibrary.com]

instruments to measure and record physiological variables. The wearer's blood pressure and heart rate were measured and recorded every 15 min for 6 years. The values of variable of interest namely mean diastolic blood pressure (DBP) are summarized in Table IX. To explore the effect of measurement error on the rate of false alarm, using error-free data, the *UCL* is set such that $ARL_0 = 200$ and the parameters of Eq. ((12)) are assumed to be as $a = 0, b = 1, \varepsilon_t \sim N(0, 1); t = 1, \ldots, 24$. The *CLR*(without error) and CLR'(with error) statistics corresponding to all 24 observations are plotted in Figure 3. Figure 3 shows that all 24 samples are in-control when there is no measurement error whereas in the presence of measurement error, the control chart triggers a false alarm in ninth sample taken.

7. Conclusion

In this paper, a new approach based on CUSUM procedure and generalized likelihood ratio statistic for joint monitoring of the process mean and variance was presented in which simulation was employed for computing some RL-based criteria. For all methods, the *UCL* value was set such that we have a same in-control ARL value in order to achieve a fair comparison. The results indicated the satisfactory ability of the proposed control chart in detecting all step shifts rather than GLR statistic. Then, the effect of measurement errors in two cases of error variance including constant and linearly increasing variance on the ability of the proposed CUSUM-based chart was studied. The results showed the adverse effects of measurement errors especially when the variance of error term was considered as linearly increasing-type. Furthermore, in order to decrease the effects of gauge measurement errors, taking several measurements on each sample point was suggested as an effective remedial approach. The obtained results supported the claim that as the number of measurement on each sample point increases, the *ARL*s, *SDRL*s, *MRL*s, and the values of Q_1 and Q_3 decreases in the both cases of constant and linearly increasing variance for error term. Finally, the applicability of the proposed approach was illustrated by a real data example. As a direction for the future researches, the proposed method can be extended for a process with multivariate quality characteristics in which the process mean vector and variance–covariance matrix should be monitored jointly. Furthermore, it could be interesting to monitor mean and variance of a process with profile quality characteristics in the presence of measurement errors.

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